Second-order condition for unconstrained optimization

Given the following function, analyze the relative extrema and, if they exist, verify the second-order condition using the Hessian and the second differential.

$$f(x, y) = x^2 + y^2$$

Solution

Finding the Critical Points

We calculate the first-order partial derivatives:

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$$

We set them equal to zero to find the critical points:

$$2x = 0 \implies x = 0, \quad 2y = 0 \implies y = 0$$

Therefore, the only critical point is:

$$(x_0, y_0) = (0, 0)$$

Calculating the Second Differential

The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

The second differential at $(0, 0)$ is:

$$d^2 f = \frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x \partial y} dx\, dy + \frac{\partial^2 f}{\partial y^2}(dy)^2$$

Substituting the obtained values:

$$d^2 f = 2(dx)^2 + 2 \cdot 0 \cdot dx\, dy + 2(dy)^2 = 2(dx)^2 + 2(dy)^2$$

Analyzing the Positivity of the Second Differential

We observe that $(dx)^2 \geq 0$ and $(dy)^2 \geq 0$ for all real values of dx and dy. Therefore:

$$d^2 f = 2\left((dx)^2 + (dy)^2\right) \geq 0$$

Moreover, $d^2 f = 0$ if and only if $dx = dy = 0$. Therefore, the second differential is **positive definite** at $(0, 0)$.

Conclusion

Since the second differential $d^2 f$ is positive definite at the critical point $(0, 0)$, we conclude that this point is a **relative minimum** of the function $f(x, y) = x^2 + y^2$.

Using the Hessian

The Hessian matrix at $(0, 0)$ is:

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Calculating the Determinant of the Hessian

The determinant of the Hessian is:

$$\det(H) = f_{xx} f_{yy} - (f_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$$

Applying the Second Derivative Test

We apply the Hessian matrix test to classify the critical point:

- If $\det(H) > 0$ and $f_{xx} > 0$, then $(0,0)$ is a **relative minimum**.
- If $\det(H) > 0$ and $f_{xx} < 0$, then $(0,0)$ is a **relative maximum**.
- If $\det(H) < 0$, then $(0,0)$ is a **saddle point**.

  In our case:

$$\begin{aligned} \det(H) &= 4 > 0 \\ f_{xx} &= 2 > 0 \end{aligned}$$

  Therefore, the critical point $(0,0)$ is a **relative minimum**.

Conclusion

Using the Hessian, we have reached the same conclusion: the function $f(x,y) = x^2 + y^2$ has a relative minimum at the point $(0,0)$.